Math 10260 Exam 2 Solutions – Fall 2012.

1. This series is not initially a geometric series, but if we write

$$\sum_{n=0}^{\infty} \frac{2^n + (-1)^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty} \left(\frac{-1}{3}\right)^n,$$

then the two terms on the right hand side are both geometric series with the form $a\sum_{n=0}^{\infty} r^n$ with $|r| < 1$ (and $a = 1$), and so they both converge to $\frac{a}{1-r}$. Therefore

$$\sum_{n=0}^{\infty} \frac{2^n + (-1)^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty} \left(\frac{-1}{3}\right)^n = \frac{1}{1-(2/3)} + \frac{1}{1-(-1/3)} = 15/4.$$

2. Since we're asked to find a sum and it doesn't look like this is going to be geometric, we hope that it is a telescoping series. We compute the partial sum s_M:

$$\begin{aligned} s_M &= \sum_{n=1}^{M} \left[\frac{5n}{n+3} - \frac{5(n+1)}{n+4}\right] \\ &= \left[\frac{5}{4} - \frac{10}{5}\right] + \left[\frac{10}{5} - \frac{15}{6}\right] + \left[\frac{15}{6} - \frac{20}{7}\right] + \cdots + \left[\frac{5M}{M+3} - \frac{5(M+1)}{M+4}\right] \\ &= \frac{5}{4} - \frac{5(M+1)}{M+4}, \end{aligned}$$

since the terms telescope. Then since a series converges to L if the sequence of partial sums converge to L, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \left[\frac{5n}{n+3} - \frac{5(n+1)}{n+4}\right] &= \lim_{M\to\infty} s_M = \lim_{M\to\infty} \sum_{n=1}^{M} \left[\frac{5n}{n+3} - \frac{5(n+1)}{n+4}\right] \\ &= \lim_{M\to\infty} \frac{5}{4} - \frac{5(M+1)}{M+4} \\ &= \frac{5}{4} - \lim_{M\to\infty} \frac{5+1/M}{1+4/M} = \frac{5}{4} - 5 = \frac{-15}{4}. \end{aligned}$$

3. (I): This series has positive terms, and so we do a limit comparison test with $\sum \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{\frac{3n^3+2n+1}{2n^5+n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{3n^5+2n^3+n^2}{2n^5+n^2} = \lim_{n\to\infty} \frac{3+2/n^2+1/n^3}{2+1/n^3} = \frac{3}{2}.$$

Since this limit is a finite number that is bigger than zero and $\sum \frac{1}{n^2}$ converges (p-series with $p = 2$), the series in (I) converges.

(II): Notice that the terms being added in the series don't go to 0:

$$\lim_{n\to\infty} \frac{n}{\ln n} \overset{\text{L'Hospital}}{=} \lim \frac{1}{1/n} = \infty.$$

Therefore, (II) diverges by the test for divergence.

(III): Ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{2^{n+2}}{3((n+1)!)}}{\frac{2^{n+1}}{3(n!)}}\right| = \lim_{n\to\infty}\left|\frac{2^{n+2}}{3((n+1)!)}\cdot\frac{3(n!)}{2^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{2}{n+1}\right| = 0 < 1,$$

so the series converges by the ratio test.

4. The series is bounded as follows

$$\sum_{n=1}^{\infty}\frac{\sin(n^2)}{n^2} \leq \sum_{n=1}^{\infty}\frac{|\sin(n^2)|}{n^2} \leq \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

The last series above converges, as it is a p-series with $p = 2$. Therefore, by the comparison test this series is absolutely convergent.

5. The series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$ is conditionally convergent. Observe, that

(1) It is alternating.

(2) The $\lim_{n\to\infty}\frac{(-1)^n}{\sqrt{n+1}} = 0$.

(3) We have the absolute value of the sequence defining the series is decreasing:

$$\frac{1}{\sqrt{(n+1)+1}} = \frac{1}{\sqrt{n+2}} \leq \frac{1}{\sqrt{n+1}}.$$

However, $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{\sqrt{n+1}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}}$, which is divergent by the Limit Comparison Test with $b_n = 1/\sqrt{n}$.

6. Using the hint, we write

$$\frac{2x}{(1-x^2)^2} = \frac{d}{dx}\left[\frac{1}{1-x^2}\right].$$

We notice that the series on the right hand side is the geometric series with $r = x^2$ therefore, we have

$$\frac{2x}{(1-x^2)^2} = \frac{d}{dx}\left[\sum_{n=0}^{\infty}x^{2n}\right] = \sum_{n=1}^{\infty}2nx^{2n-1}.$$

7. We know that e^x has power series representation $\sum_{n=0}^{\infty}\frac{x^n}{n!}$. So we substitue x for $\frac{3}{5}$ and we get $e^{3/5} = \sum_{n=0}^{\infty}\frac{(\frac{3}{5})^n}{n!}$. So $2e^{3/5} = 2\sum_{n=0}^{\infty}\frac{(\frac{3}{5}^n)}{n!} = \sum_{n=0}^{\infty}\frac{2\cdot 3^n}{5^n(n!)}$.

8. The fouth order Taylor polynomial is given by $\sum_{n=0}^{4}\frac{f^{(n)}(a)}{n!}(x-a)^n$. So we know the $(x-3)^2$ coefficient must be $\frac{f^{(2)}(3)}{2!}$. Thus:

$$\frac{f^{(2)}(3)}{2!} = 10 \implies f^{(2)}(3) = 10\cdot 2! = 20.$$

9. We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{2x^{n+1}}{3^{n+1}(n+1)^2}\frac{3^n n^2}{2x^n}\right| = \lim_{n\to infty}\left|\frac{xn^2}{3(n+1)^2}\right| = \left|\frac{x}{3}\right|\lim_{n\to\infty}\frac{n^2}{(n+1)^2} = \left|\frac{x}{3}\right|$$

The ratio test tells us this series converges when the limit we just calculated is less than 1. So we have that $\left|\frac{x}{3}\right| < 1 \implies |x| < 3$. Thus $R = 3$.

10. The power series of $\sin(x)$ is $\sin(x) = x - \frac{1}{6}x^3 + \frac{1}{5!}x^5 - \cdots$, and therefore the power series of $\sin(x^{10})$ is $x^{10} - \frac{1}{6}x^{30} + \frac{1}{5!}x^{50} - \cdots$. Accordingly, we are finding

$$\lim_{x\to 0}\frac{-\frac{1}{6}x^{30} + \frac{1}{5!}x^{50} - \cdots}{x^{30}} = \lim_{x\to 0}(-\frac{1}{6} + \frac{1}{5!}x^{20} - \cdots) = -\frac{1}{6}.$$

11. First we must verify that the test is applicable. Setting $f(x) = \frac{1}{x\ln(x)}$, the series is given by $\sum_{2}^{\infty} f(n)$. Clearly this function is continuous and positive on $[2,\infty)$, and since both x and $\ln(x)$ are increasing, $f(x)$ is decreasing. Now we need to find the improper integral $\int_2^\infty \frac{1}{x\ln(x)}\mathrm{d}x$.

Letting $u = \ln(x)$ we have $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$, so that the integral becomes

$$\int_2^\infty \frac{1}{x\ln(x)}\mathrm{d}x = \lim_{t\to\infty}\int_2^t \frac{1}{x\ln(x)}\mathrm{d}x = \lim_{t\to\infty}\int_{\ln(2)}^{\ln(t)}\frac{1}{u}\mathrm{d}u = \lim_{t\to\infty}\ln(u)|_{\ln(2)}^{\ln(t)} = \lim_{t\to\infty}\ln(\ln(t)) - \ln(\ln(2)) = \infty.$$

Therefore the series diverges.

12. (a) The Taylor series expansion of $\cos(x)$ is easily calculated by repeatedly differentiating $\cos(x)$, and is given by $\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{6!}x^6 + \cdots + (-1)^n\frac{1}{(2n)!}x^{2n} + \cdots$.

(b) To find the series expansion of $\cos(x^2)$ we simply replace x with x^2 to get $\cos(x^2) = 1 - \frac{1}{2}x^4 + \frac{1}{24}x^8 - \frac{1}{6!}x^{12} + \cdots + (-1)^n\frac{1}{(2n)!}x^{4n} + \cdots$.

(c) We now want to find the definite integral $\int_0^{0.1}\cos(x^2)\mathrm{d}x$. We do this by integrating the series term by term so that

$$\begin{aligned}\int_0^{0.1}\cos(x^2)\mathrm{d}x &= \int_0^{0.1}(1 - \frac{x^4}{2} + \frac{x^8}{24} - \frac{x^{12}}{6!} + \cdots + (-1)^n\frac{x^{4n}}{(2n)!} + \cdots)\mathrm{d}x \\ &= (x - \frac{x^5}{10} + \frac{x^9}{216} - \frac{x^{13}}{13\cdot 6!} + \cdots + (-1)^n\frac{x^{4n+1}}{(4n+1)(2n)!} + \cdots)|_0^{0.1} \\ &= 0.1 - \frac{(0.1)^5}{10} + \frac{(0.1)^9}{216} - \frac{(0.1)^13}{13\cdot 6!} + \cdots + (-1)^n\frac{(0.1)^{4n+1}}{(4n+1)(2n)!} + \cdots.\end{aligned}$$

(d) Because this is a decreasing alternating series, we can estimate the integral to within 10^{-8} by finding the first term which is smaller than 10^{-8}, and only summing the terms which precede it. This is the third term, $\frac{(0.1)^9}{216}$. Our estimate is $0.1 - \frac{(0.1)^5}{10} = .099999$.

13. To find the radius of convergence, lets use the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-4)^{n+1}}{2^{(n+1)}((n+1)+1)}\cdot\frac{2^n(n+1)}{(-1)^n(x-4)^n}\right| = \lim_{n\to\infty}\left|\frac{(x-4)}{2}\cdot\frac{n+1}{n+2}\right| = \frac{|x-4|}{2}.$$

By Ratio Test, the series will converge if $\frac{|x-4|}{2} < 1$, this is if $|x-4| < 2$. Hence, the radius of convergence is 2.

Now we need to verify convergence in the endpoints $x = 2$ and $x = 6$. When $x = 2$, $\sum_{n=0}^{\infty}\frac{(-1)^n(2-4)^n}{2^n(n+1)} = \sum_{n=0}^{\infty}\frac{1}{(n+1)}$, which diverges by the Integral Test. At $x = 6$, $\sum_{n=0}^{\infty}\frac{(-1)^n(6-4)^n}{2^n(n+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)}$, which converges by the Alternating Series Test. Hence, the interval of convergence is $(2, 6]$.